DOCUMENT RESUME

ED 187 593 SE 031 112

TITLE	The Arithmetic Project Course for Teachers - 12. Topic: More Work With Artificial Operations. Supplement: Functions.
INSTITUTION	Education Development Center, Inc., Newton, Mass.: Illinois Univ., Urbana.
SPONS AGENCY	Carnegie Corp. of New York, N.Y.: National Science Foundation, Washington, D.C.
PUB DATE	73
NOTE	37p.: For related documents, see SE 031 100-120. Contains occasional light type.
EDRS PRICE	MF01/PC02 Plus Postage.
DESCRIPTORS	Algorithms; Elementary Education; *Elementary School Mathematics; *Elementary School Teachers; Films; Grade 5; Inservice Education; *Inservice Teacher Education; Mathematics Curriculum; *Mathematics Instruction; Mathematics Teachers; Number Concepts; *Problem Sets; Teacher Education
IDENTIFIERS	*University of Illinois Arithmetic Project

ABSTRACT

This is one of a series of 20 booklets designed for participants in an in-service course for teachers of elementary mathematics. The course, developed by the University of Illinois Arithmetic Project, is designed to be conducted by local school personnel. In addition to these booklets, a course package includes films showing mathematics being taught to classes of children, extensive discussion notes, and detailed guides for correcting written lessons. This booklet contains exercises on work with artificial operations, a summary of the problems in the film "Frames and Number Line Jumping Rules," and the supplement. (MK)

THE ARITHMETIC PROJECT COURSE FOR TEACHERS

TOPIC: More Work With Artificial Operations.

FILM: Frames and Number Line Jumping Rules, Grade 5

SUPPLEMENT: Functions

NAME: ....

12

This booklet is part of a course for teachers produced by The Arithmetic Project in association with Education Development Center. Principal financial support has come from the Carnegie Corporation of New York, the University of Illinois, and the National Science Foundation.

The course is available from:

THE ARITHMETIC PROJECT
Education Development Center
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Newton, Massachusetts 02160

BOOK TWELVE

1.

Here is an operation that is used in some branches of mathematics. We shall use its conventional name, "delta", symbolized by the Greek letter δ. Here are some examples of how it works:

$$3 \ \delta \ 7 = 0 \qquad -5\tfrac{1}{2} \ \delta \ -5\tfrac{1}{2} = 1$$

$$7 \ \delta \ 3 = 0 \qquad -5\tfrac{1}{2} \ \delta \ 5\tfrac{1}{2} = 0$$

$$7 \ \delta \ 7 = 1 \qquad 3\tfrac{7}{8} \ \delta \ 3\tfrac{3}{4} = 0$$

$$\tfrac{4}{2} \ \delta \ 2 = 1$$

What does δ do? (See page 12 to confirm your answer.)

In some of the problems that follow it may be that more than one number works. In each case describe all the numbers that work.

1. $14 \ \delta \ 13 =$ ________

2. $14 \ \delta \ 14 =$ ________

3. $-6\frac{2}{3} \ \delta \ \square = 1$

4. $-6\frac{2}{3} \ \delta \ \square = 0$

5. $-6\frac{2}{3} \ \delta \ \square = 2$

6. $(3 \; \delta \; 4) \; \delta \; 5 = \square$

7. $3 \; \delta \; (4 \; \delta \; 5) = \square$

8. Find a number for $\square$ so that $(\square \; \delta \; 4) \; \delta \; 5$ does not equal $\square \; \delta \; (4 \; \delta \; 5)$.

9. $(5 \; \delta \; 8) + (5 \; \delta \; 5) =$ __________

10. $(6 \; \delta \; 8) + (6 \; \delta \; 5) =$ __________

11. $(\square \; \delta \; 3) + (\square \; \delta \; 7) = 1$

12. $(\square \; \delta \; 3) + (\square \; \delta \; 7) = 0$

13. $(\square \; \delta \; 3) + (\square \; \delta \; 7) = 2$

14. $(\square \; \delta \; 3) + (\square \; \delta \; 7) + (\square \; \delta \; 40\frac{1}{2}) = 1$

15. Make up a single equation in which 1, 3, $8\frac{1}{4}$, 14, and 101 work in the boxes, and in which only those numbers work.

Here are some examples of an operation which we shall call "check": *

$$6 \checkmark 10 = 2$$

$$50 \checkmark 60 = 5$$

$$59 \checkmark 19 = 20$$

$$380 \checkmark 379 = \frac{1}{2}$$

$$11 \checkmark 11 = 0$$

1. How might a student describe what check does?

2. If you can, give a formal definition of check. (If you have trouble with this question, consult page 12.)

$$\square \checkmark \triangle \stackrel{df}{=}$$

Now do these:

3. $7 \checkmark 15 =$ __________

4. $22 \checkmark 29 =$ __________

5. $29 \checkmark 22 =$ __________

* In a previous lesson, ✓ meant something else. We use the symbol again but with a different meaning to emphasize that this choice is arbitrary. You do need to be sure that your students know what a particular symbol means on a given day.

Continue with: $\square \checkmark \triangle \stackrel{df}{=} \frac{|\square - \triangle|}{2}$

6. $1{,}076 \checkmark 1{,}070$ = ____________

7. $1{,}076\frac{2}{3} \checkmark 1{,}070\frac{2}{3}$ = ____________

8. $0 \checkmark -6$ = 3

9. $0 \checkmark -11$ = ____________

10. $0 \checkmark 11$ = ____________

11. $-2 \checkmark 14$ = 8

12. $-2\frac{1}{2} \checkmark 14\frac{1}{2}$ = ____________

13. $-300 \checkmark -400$ = ____________

(The answer is not -50.)

14. $-200 \checkmark 200$ = ____________

What is a likely wrong answer? ____________

15. $\frac{1}{19} \checkmark \frac{5}{19}$ = ____________

16. $\frac{17}{25} \checkmark 1$ = ____________

17. $\frac{17}{25} \checkmark 5$ = ____________

Continue with: $\square \checkmark \triangle \overset{\text{df}}{=} \dfrac{|\square - \triangle|}{2}$

18. $33 \checkmark \square = 5$

19. A second answer to 18 above: ____________

Find two answers each for problems 20, 21, and 22.

20. $21 \checkmark \square = \frac{1}{2}$ ____________, ____________

21. $100 \checkmark \square = 12\frac{1}{4}$ ____________, ____________

22. $\square \checkmark 57 = 100$ ____________, ____________

23. $6 \checkmark 18 = \square$

24. $\square \checkmark 15 = \square$

25. $\square \checkmark 33 = \square$

26. $\square \checkmark 81 = \square$

27. $\square \checkmark \triangle = 0$ What are all the numbers that work?

Continue with: $\square \checkmark \triangle \stackrel{df}{=} \dfrac{|\square - \triangle|}{2}$

28. $(5 \checkmark 20) + (20 \checkmark 35) =$ __________

29. $(5 \checkmark 21\frac{1}{2}) + (21\frac{1}{2} \checkmark 35) =$ __________

30. $(5 \checkmark 31\frac{5}{8}) + (31\frac{5}{8} \checkmark 35) =$ __________

31. $(10 \checkmark \square) + (\square \checkmark 90) = 10 \checkmark 90$

What are all the numbers that work?

32. Give one number that does not work in problem 31: __________

III.

The operations we have called delta and check (as well as star, circle-dot, and check used in the previous written lesson) are binary operations. You have to give them <u>two</u> numbers to get back one number. Addition, subtraction, multiplication, and division are also binary operations, but we are too familiar with them to appreciate their general properties. By working with check and other artificial operations we shall eventually learn more about the way familiar operations work.

We shall now explore whether these and other such operations are commutative.

When an operation such as circle-dot* is commutative, switching the two numbers around never changes the answer. Examples:

$$3 \odot 5 = 5 \odot 3 \quad ; \quad 11\tfrac{1}{2} \odot -2 = -2 \odot 11\tfrac{1}{2} \quad ; \quad \text{and so on.}$$

If you can find two numbers which give different answers when you switch them, then the operation is not commutative.

In the following problems new operations are defined. Tell whether each operation is commutative. If it is not commutative, give an example to show this, as in the first two illustrations. See if you can predict commutativity or non-commutativity before trying specific numbers.

Commutative?

1. $\square \otimes \triangle \stackrel{df}{=} \square + 3 \times \triangle$ NO

If not commutative, illustrate by giving an example.

$1 \otimes 5 = 16$

but $5 \otimes 1 = 8$

2. $\square \,\mathrm{I}\, \triangle \stackrel{df}{=} \max\left(\left|\square\right|, \triangle\right)$ NO

If not commutative, illustrate by giving an example.

$3 \,\mathrm{I}\, -10 = 3$

but $-10 \,\mathrm{I}\, 3 = 10$

---

* $\square \odot \triangle \stackrel{df}{=} \dfrac{\square + \triangle}{2}$

Commutative?

3. $\square \odot \triangle \overset{df}{=} \dfrac{\square + \triangle}{2}$ ________

If not commutative, illustrate by giving an example.

4. $\square * \triangle \overset{df}{=} 2 \times \square + \triangle$ ________

If not commutative, illustrate by giving an example.

5. $\square \checkmark \triangle \overset{df}{=} \dfrac{|\square - \triangle|}{2}$ ________

If not commutative, illustrate by giving an example.

6. $\square \,\text{!}\, \triangle \overset{df}{=} \square \times 3 \times \triangle$ ________

If not commutative, illustrate by giving an example.

7. $\square \top \triangle \stackrel{\text{df}}{=} \square - \triangle - \square + \triangle$ ______

If not commutative, illustrate by giving an example.

8.[†] $\square \# \triangle \stackrel{\text{df}}{=} (\square - \triangle) - (\square + \triangle)$ ______

If not commutative, illustrate by giving an example.

9. $\square$ ⚓ $\triangle \stackrel{\text{df}}{=} \square \times \triangle + \square + \triangle$ ______

If not commutative, illustrate by giving an example.

---

† Observe that problems 7 and 8 are very different.

For the following problems, $\odot$, $\surd$, δ, and $*$ are defined as before in this lesson:

$$\square \odot \triangle \ \stackrel{df}{=} \ \frac{\square + \triangle}{2}$$

$$\square \surd \triangle \ \stackrel{df}{=} \ \frac{|\square - \triangle|}{2}$$

$$\square * \triangle \ \stackrel{df}{=} \ 2 \times \square + \triangle$$

$$\square \ \delta \ \triangle \ \stackrel{df}{=} \ \begin{cases} 0 & \text{if } \square \neq \triangle \\ 1 & \text{if } \square = \triangle \end{cases}$$

Commutative?

☆10. $\square \ \$ \ \triangle \ \stackrel{df}{=} \ |\square \odot \triangle|$ __________

If not commutative, illustrate by giving an example.

☆11. $\square \ @ \ \triangle \ \stackrel{df}{=} \ |\square| \odot \triangle$ __________

If not commutative, illustrate by giving an example.

Continue with the definitions:

$$\square \odot \triangle \ \stackrel{\text{df}}{=} \ \frac{\square + \triangle}{2}$$

$$\square \checkmark \triangle \ \stackrel{\text{df}}{=} \ \frac{|\square - \triangle|}{2}$$

$$\square * \triangle \ \stackrel{\text{df}}{=} \ 2 \times \square + \triangle$$

$$\square \ \delta \ \triangle \ \stackrel{\text{df}}{=} \ \begin{cases} 0 & \text{if } \square \neq \triangle \\ 1 & \text{if } \square = \triangle \end{cases}$$

Find all the pairs of numbers $(\square, \triangle)$ that work for problems ☆12, ☆13, ☆14, and ☆15.

☆12. $\square \ \delta \ \triangle = \square \odot \triangle$

☆13. $\square \checkmark \triangle = \square \odot \triangle$

☆14. $\square * \triangle = \square + \triangle$

☆15. $\square * \triangle = \square$

Epilogue

A satisfactory description of what δ does (page 1) is

> If the numbers are the same, the answer is 1;
> if they are different, the answer is 0.

An acceptable answer to question 1 about $\checkmark$ on page 3 is

> Take one-half the distance between the two numbers.

An acceptable answer to question 2 might be

$$\square \checkmark \triangle \;\stackrel{\text{df}}{=}\; \frac{|\square - \triangle|}{2}$$

Another acceptable answer:

$$\square \checkmark \triangle \;\stackrel{\text{df}}{=}\; \begin{cases} \dfrac{\square - \triangle}{2} & \text{if } \square \geq \triangle \\[2ex] \dfrac{\triangle - \square}{2} & \text{if } \triangle \geq \square \end{cases}$$

* * *

As has been suggested here and in the films, a popular way of introducing an artificial operation to a class of children is to give a variety of examples and let students guess what the operation is by supplying answers to problems. In time students may be asked to give a definition of the operation in words or symbols.

Beware, however, of carrying such guessing too far. If in a reasonable time your students have not figured out what you have in mind, it is best to tell them and proceed to explore the properties of the operation.

Even after a number of examples, one could invent valid (if fancy) definitions fitting the examples given but not the same as the one the teacher was thinking of. After the first three examples on page 3, a child would be correct who said that the operation "check" is as follows:

> Add the two numbers. Round to the nearest multiple of 10. Divide by 4. Add 1. Again round to the nearest multiple of 10. If the first digit is odd, divide by 5. Then, if the first digit is not 2, subtract 1.

This definition does not work for the next example on page 3—it gives 37 instead of $\frac{1}{2}$—but an operation could be invented which would work for that example as well and similarly for any number of examples.

Notice also that the examples given on page 3 do not tell you what to do with negative numbers—this is given by some answered questions later on (problems 8 and 11).

Summary of Problems in the Film
"Frames and Number Line Jumping Rules"

5th Grade, Cunniff School, Watertown, Massachusetts
Teacher: Lee Osburn

Here are some problems:

□ + □ + □ = 21	(7)
□ + □ + □ = 24	(8)
□ + □ + □ = 22	(Written between the two problems above) (Wrong answer: 10 + 10 + 2)

What is the difference between these two problems:

□ + □ + △ = 22

and □ + □ + □ = 22

Is the answer to □ + □ + □ = 22 larger or smaller than 7? 8?

□ + □ + □ = 23 (Written after □ + □ + □ = 22)

More problems:

□ + □ + □ = 60	(20)
□ + □ + □ = 6[illegible]	($20\frac{1}{3}$)
□ + □ + □ = 91	($30\frac{1}{3}$)

New problem:

□ + □ + □ = □ + 12 (Wrong answer: 3)

Let's do □ + □ + □ + □ = 12

Continue with the problems:

□ + □ + □ = □ + 20

□ + □ + □ = □ + 2

□ + □ + □ = □ + 50

□ + □ + □ = □ + 18

□ + □ + □ = □ + 30

□ + □ + □ = □ + 40

I want a number so that the answer in the boxes will come out to be 75.

□ + □ + □ = □ + ____ (150)

What is the answer so that the number in the boxes is $10\frac{1}{2}$? (21)

"Andy, I notice your hand is up awfully fast, do you have a method for this?"

"Yes, the first two numbers in the first two boxes have to add up to the number on the right."

More problems:

□ + □ + □ + □ = □ + 15 (Wrong answer: $7\frac{1}{2}$)

□ + □ + □ + □ = □ + 21

□ + □ + □ + □ = □ + 24

□ + □ + □ + □ = □ + 22

□ + □ + □ + □ = □ + 23

New problems:

□ + □ + □ = 300

99 + 99 + 99 = △

98 + 98 + 98 = ⏢

96 + 96 + 96 = ▭

Did you actually add these numbers? How did you get the answer?

New problem:

4 + □ + □ + 3 = (2 × □) + 7 (All numbers work.)

"This is a rule which tells you how to make jumps on a line:"

$$\boxed{} \longrightarrow 3 \times \boxed{} - 10$$

Andy: "10"

"What about 10?"

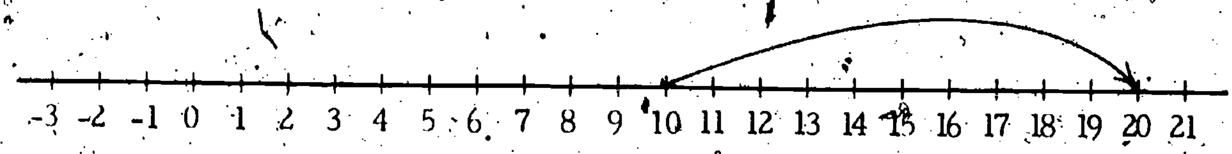
Andy: "That's what goes in there, isn't it?"

"It might. This is a rule that tells us that we are going to start here, and land here. So let's start at 10. If we start at 10, where do we land, Gretchen?"

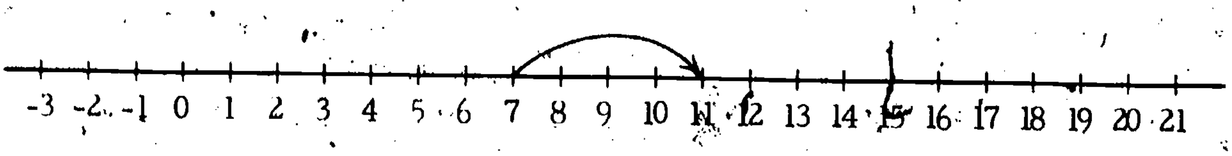
(30)

"That's the first part of it. Now do the whole thing."

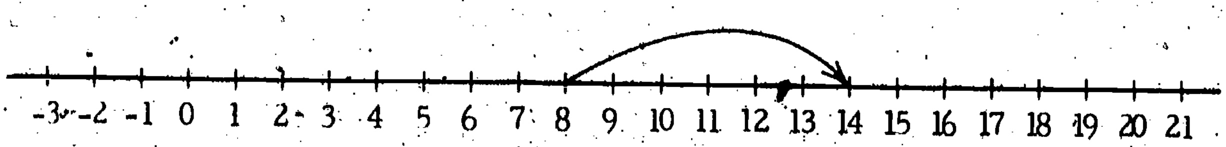
(20)



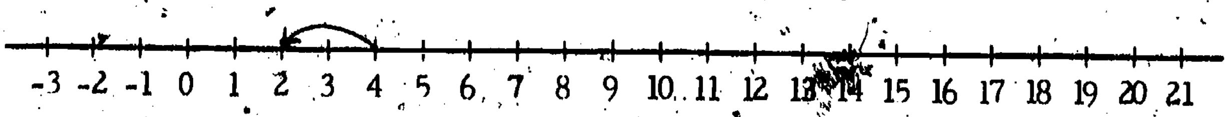
Start at: 7 Land at: 11



Start at: 8 Land: 14



Start: 4 Land: 2



Continue with the rule $\square \longrightarrow 3 \times \square \div 10$:

What happened to that jump? Is that any different from the other one? Lynn?

(It's going the opposite way.)

Let's see if we can find some more jumps going the opposite way. You give me a number.

(17)

If you start at 17 you go to 41. What if you start at 3? Land?

(-1)

How many spaces is the jump that starts at 4? at 3? at 7?

We have two jumps that are 4 spaces each. Can you find another jump of 2 spaces?

(Start at 6)

Can you find a jump that doesn't go anywhere?

Can you find a starting point so that we will jump one space?

Supplement

Functions

by Edward Esty

The University of Illinois Arithmetic Project has long used jumping rules in classes for elementary school children. This paper will give a broader view of jumping rules (called "functions" by mathematicians) and indicate how they are used in later work in mathematics.

Example I. Consider the jumping rule $\Box \longrightarrow \Box + 3$

(This is a standard introductory example from the Arithmetic Project.) If we start at 4, we put a 4 in both of the boxes. The rule then tells us to go from 4 to 4 + 3 or 7.

$$\underbrace{\Box}_{\text{Start}} \longrightarrow \underbrace{\Box + 3}_{\text{Land}}$$

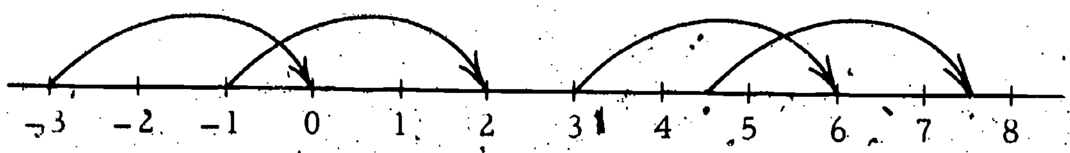
Also, if we start at 2, we land on 5. If we start at 100, we land on 103, and so on.

There are several things which should be noticed about this jumping rule (or function).

1. There are some places where we <u>can't</u> start. If we put Bronx Zoo into the boxes, we get nonsense: Bronx Zoo + 3 doesn't mean anything.

2. If we start at a "legal" starting place, we land on exactly one landing place. In the absence of further information we can assume that "legal" starting points include at least all the numbers on the number line—zero, positive and negative integers, all fractions (like $-\frac{3}{4}$, $6\frac{1}{8}$, $\frac{22}{7}$, etc.) and even irrational numbers like $\sqrt{2}$ and π.

3. We can draw a picture of some of the jumps made with the jumping rule. Typically, we might draw:



This jumping rule, $\square \rightarrow \square + 3$, is not an overly fascinating one but we want a simple one for illustration now. Notice that <u>not all possible jumps are drawn</u>.

4. There is a definite, simple pattern apparent in the jumps; <u>i.e.</u> each jump for this rule goes three units to the right.

5. We can land at any number we want (just by starting 3 units to the left of it.)

6. Any landing place is also a legal starting place.

7. You never land on the place you started from.

<u>Example II</u>. Here is a different sort of jumping rule—an alphabet jumping rule:

$\underbrace{\square}_{\text{Start}} \longrightarrow \underbrace{\text{the letter immediately following } \square \text{ in the English alphabet}}_{\text{Land}}$

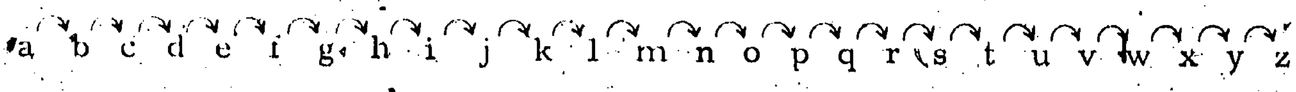
If we start at g we land at h. If we start at u we land on v, and so on.

Notice that:

1. Again there are some places we can't start at, for example $5\frac{1}{2}$, π, and z. (Why not? How could we alter the rule slightly and in a "natural" way to make z a legal starting point?)

2. As before, if we start at a legal starting place (i.e. any letter of the English alphabet except z) we land on exactly one landing place.

3. For this rule we can draw a picture of all of its jumps.



4. Again there's a simple pattern apparent in the rule.

5. We cannot land on every letter. (Which one or ones are not landing points?)

6. Not every landing place is also a legal starting place.

7. Again there are no standstill points (places such that if we start there, we land there).

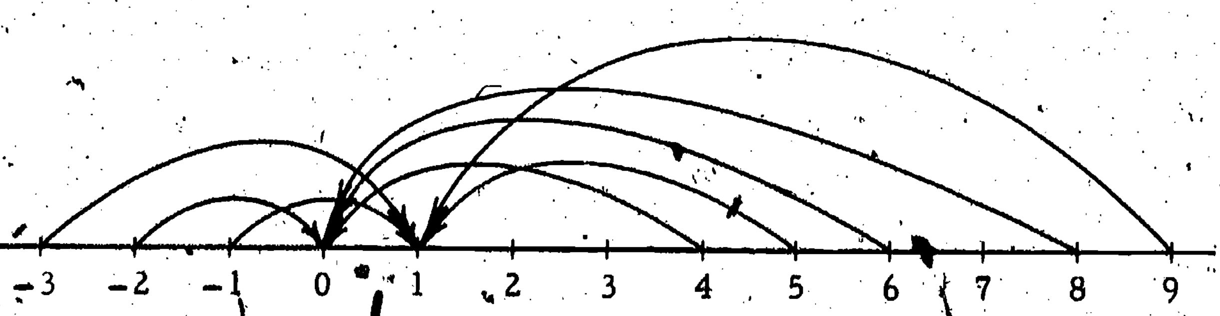
Example III. Here is another jumping rule:

$$\square \longrightarrow \begin{cases} 0 & \text{if } \square \text{ is even} \\ 1 & \text{if } \square \text{ is odd} \end{cases}$$

1. There are places we can't start at—in fact there are some numbers we can't start at (for example, $4\frac{1}{3}$).

2. If we start at a legal starting place (i.e. any integer) we land on exactly one landing place (which must be either 0 or 1).

3. We cannot draw a picture of all the jumps we can make with this rule. Here's a picture of some of them:



4. Again some sort of pattern is apparent.

5. We cannot land on every number or even on every integer.

6. Every landing place is also a legal starting place.

7. This rule does have standstill points. (What are they? Zero is an even number.)

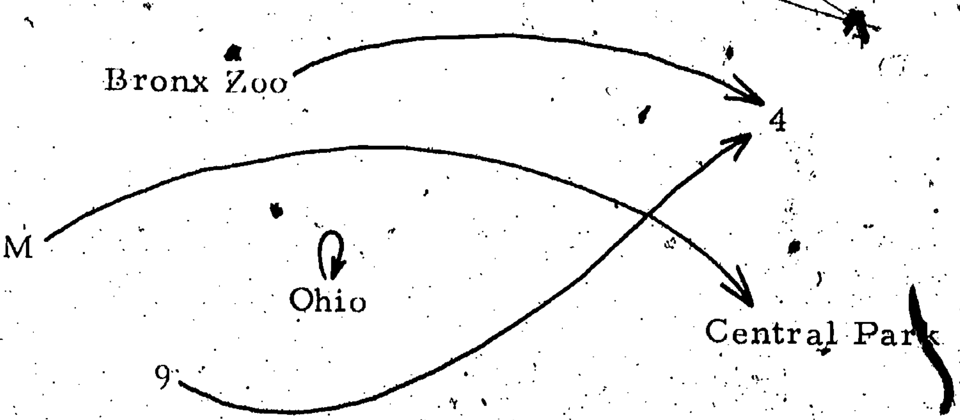
Example IV. Here is a deliberately strange jumping rule.

$$\boxed{} \longrightarrow \begin{cases} 4 \text{ if } \boxed{} \text{ is Bronx Zoo} \\ \text{Central Park if } \boxed{} \text{ is M} \\ \text{Ohio if } \boxed{} \text{ is Ohio} \\ 4 \text{ if } \boxed{} \text{ is 9} \end{cases}$$

1. Here there are only four legal starting places, namely, Bronx Zoo, M, Ohio, and 9.

2. If we start at a legal starting place we land on exactly one landing place.

3. We can draw a picture of all the jumps:



4. There is no nice pattern in this totally arbitrary and artificial jumping rule.

5. We can land on only three things, namely 4, Central Park and Ohio.

6. Not every landing point is a legal starting point.

7. This rule has a standstill "point". (Ohio)

Example V.

$$\square \longrightarrow \square$$

1. Here we shall say that there are no illegal starting places.* If you start at $5\frac{1}{2}$ you land on $5\frac{1}{2}$. If you start at Bronx Zoo, you land on Bronx Zoo, and so on.

2. If we start at a legal starting place we land on exactly one landing place.

3. We cannot draw a picture of all the jumps.

4. The pattern is clear here (and dull!)

5. We can land wherever we want.

6. Every landing place is also a legal starting place.

7. Every place is a standstill point.

The reader will have observed that there is only one property which all of the above examples share: for each legal starting point there is exactly one landing place. Some landing places may be "used" more than once (like 4 in Example IV or 0 and 1 in Example III), but to each starting point there corresponds one and only one landing point.

Example VI. Here is an example of a "rule" which is not a jumping rule (function):

$$\square \longrightarrow \text{any number whose square is } \square$$

The permissible starting numbers are zero and all positive numbers. But if we start at 9, for example, we could land on either 3 or -3, since 3^2 and $(-3)^2$ are both 9. Since for some starting numbers we have two landing numbers, this "rule" is not a function.

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*Actually, we should say that each place in any predetermined collection of starting places is a legal starting place—but don't worry about the distinction between this sentence and the one above.

Example VII.

$$\square \longrightarrow \text{the telephone number of } \square$$

We specify that the legal starting places are all people and organizations listed in the 1966 Boston West Suburban telephone directory. Suppose that if we start at H. M. Smith we land at 235-6373. (Note that one would have to have the directory to determine the landing places in general.) Is this a function? What about standstill points? Are any landing places also legal starting places?

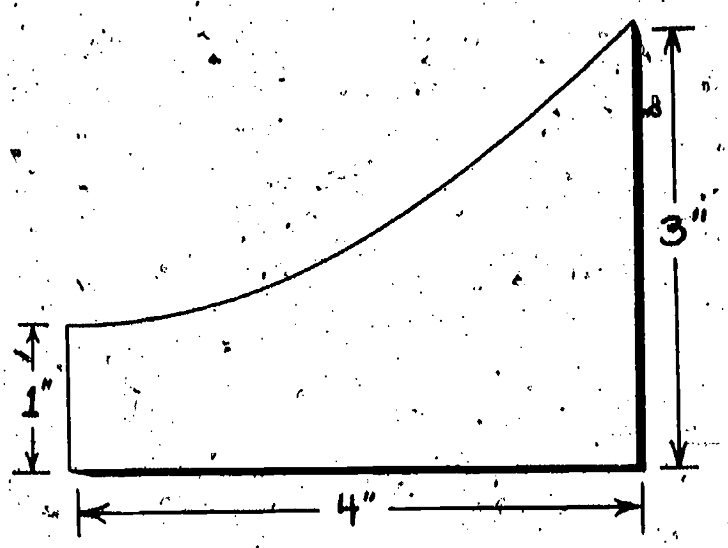
Example VIII.

$$\square \longrightarrow \begin{cases} 5 & \text{if } \square \text{ is larger than } 2 \\ \square + 3 & \text{if } \square \text{ is smaller than } 7 \end{cases}$$

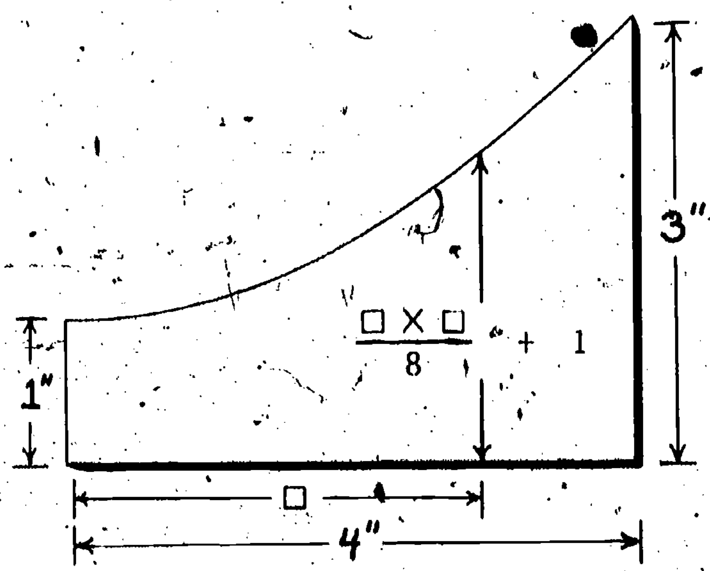
Legal starting places here are all numbers on the number line. Is this a function? Are there any standstill points?

The reader might reasonably wonder what aspects of functions people study. In the next three examples we explore in somewhat more depth the sorts of things which can be done with functions.

Example IX. Suppose we have a piece of sheet metal which looks like this:



The height of the piece obviously depends on where one measures it, but it is important to notice that given any distance (from 0 to 4 inches) from the left-hand edge, the piece of metal has exactly one height at that point. As one can see from the diagram, the height zero inches from the left-hand edge is 1 inch, and the height four inches from the left-hand edge is 3 inches. Suppose that the piece of metal is designed so that if we measure the height at a point □ inches from the left-hand edge, then the height at that point will be $\frac{\square \times \square}{8} + 1$ inches.

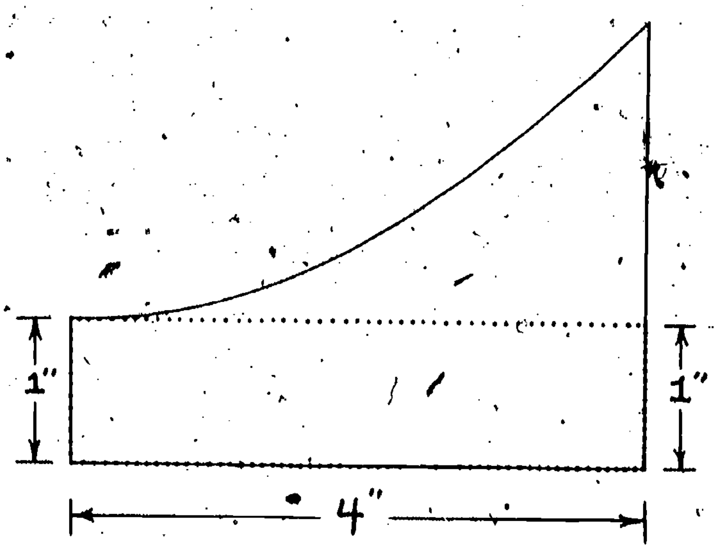


For instance, the height of the piece 2 inches from the left-hand edge is $\frac{2 \times 2}{8} + 1$ or $1\frac{1}{2}$ inches. We can express all this as a function as follows:

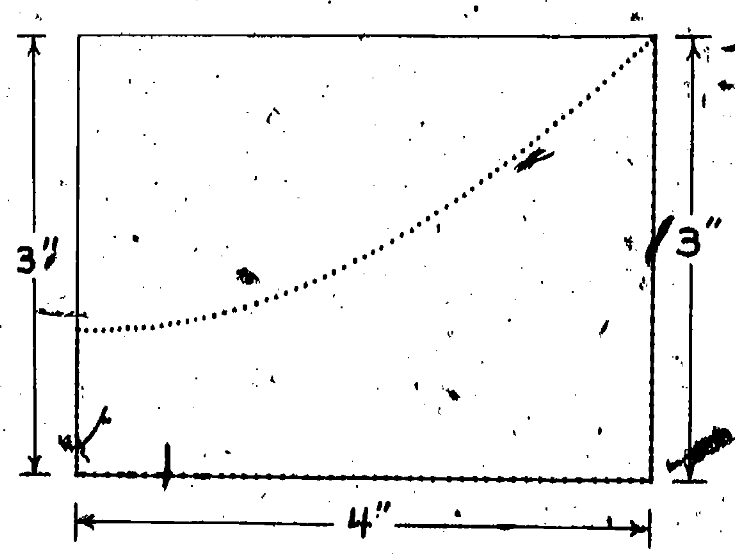
$$\square \longrightarrow \frac{\square \times \square}{8} + 1$$

The legal starting places (distances from the left-hand edge) are all the numbers from 0 to 4, inclusive; landing numbers are heights. Notice that the function gives 1 when we start at 0, and 3 when we start at 4, which agrees (as it should) with the dimensions shown in the picture.

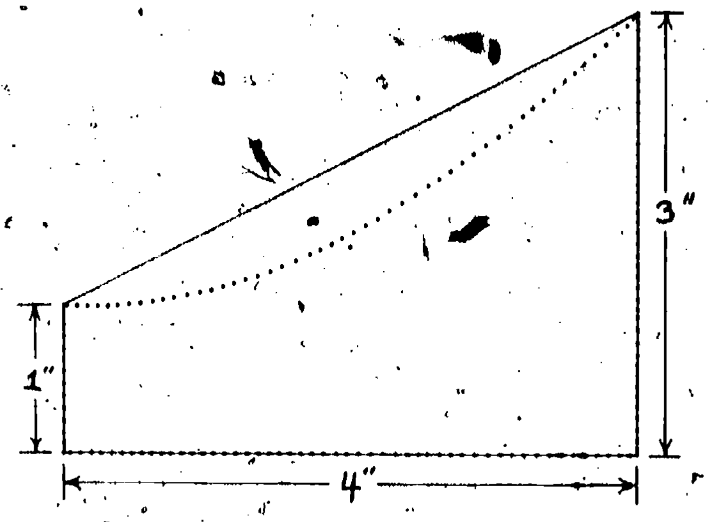
Question: What is the <u>area</u> of the front face of the piece of metal? We might <u>approximate</u> the area in many ways. Surely the area must be more than 4 square inches because a 1" X 4" rectangle could be covered by the metal.



On the other hand, the area must be less than 12 square inches since a 3" X 4" rectangle could cover the piece.

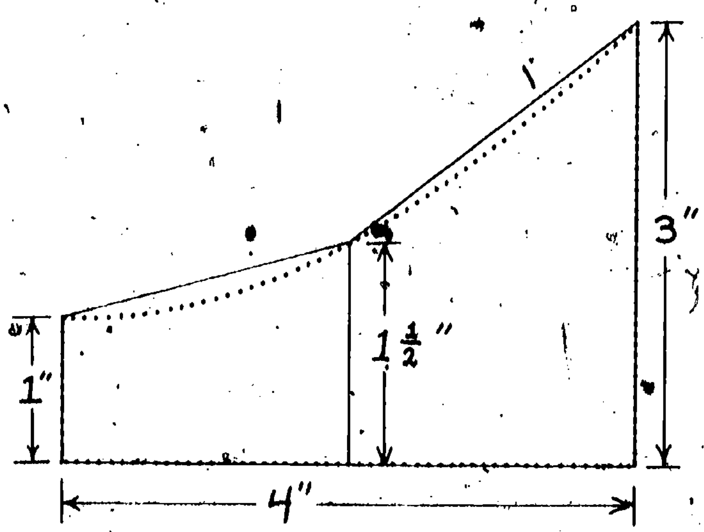


A closer approximation can be obtained by drawing a slanted line:



The area of this trapezoidal figure is $\frac{1 + 3}{2} \times 4$ or 8 square inches.*

A further refinement would be to use two trapezoids:



* If we had two of these trapezoids, they could be arranged to form a 4" by 4" square, which would have an area of 16 square inches. One of the trapezoids, then, has half that area, or 8 square inches.

We know the height of the broken vertical line in the middle because we know where we land when we start at 2 using the rule $\square \longrightarrow \frac{\square \times \square}{8} + 1$. (We've already done this. The height 2 inches from the left is $1\frac{1}{2}$ inches.) The area of the trapezoid on the left turns out to be $2\frac{1}{2}$ square inches, and the area of the one on the right is $4\frac{1}{2}$ square inches. The total area of the two trapezoids is then 7 square inches. This looks as if it is very close to the true surface area, but it is still a bit too big. We could use more and more trapezoids, but no matter how many trapezoids we use, the curved portion of the piece of metal will still give us trouble. By splitting up the figure into more and more trapezoids and by figuring the total area each time, we will get a sequence of numbers which approximate more and more closely the true area. (We are assuming here that there actually is a number which represents the true area.) Even though no single approximation will give us the true area, it is possible to determine the true area from the sequence of approximations. The following table was prepared by a computer:

Number of trapezoids	Width of each trapezoid in inches	Total area of the trapezoids in square inches
1	4	8
2	2	7
3	1.3333	6.8148
4	1	6.75
5	.8	6.72
10	.4	6.68
15	.2666	6.6726
25	.16	6.6688
50	.08	6.6672
100	.04	6.6668

Can the reader guess what number is being approximated in the right-hand column of the table? This number represents the area of the piece of metal.

There is a branch of mathematics called calculus which deals with certain properties of functions. In particular, calculus can be used to answer just such problems as the one above—finding the surface area of the piece of metal rapidly and easily.

The function which determines the curved portion of the piece of metal is:

$$\square \longrightarrow \frac{\square \times \square}{8} + 1$$

Using a process called integration (which is very much like our summing of areas of smaller and smaller trapezoids) we get a new function:

$$\square \longrightarrow \frac{\square \times \square \times \square}{24} + \square$$

which tells us the amount of area to the left of the vertical line $\square$ inches from the left-hand edge. (As before, the legal starting places are the numbers from 0 to 4, inclusive.) So if we start at 4 and use the area rule, the number we land on will be the area of everything to the left of the vertical line 4 inches from the left-hand edge—that is, the area of the whole thing. Try it. Does your answer agree with your previous guess?

The preceding paragraph, especially the part about the new function, may seem like black magic to the reader. It isn't really, but a derivation of the new function is too long to include here. Suffice it to say that often one function is useful to describe certain properties of another.

<u>Example X.</u> We reconsider Example III in a little more depth.

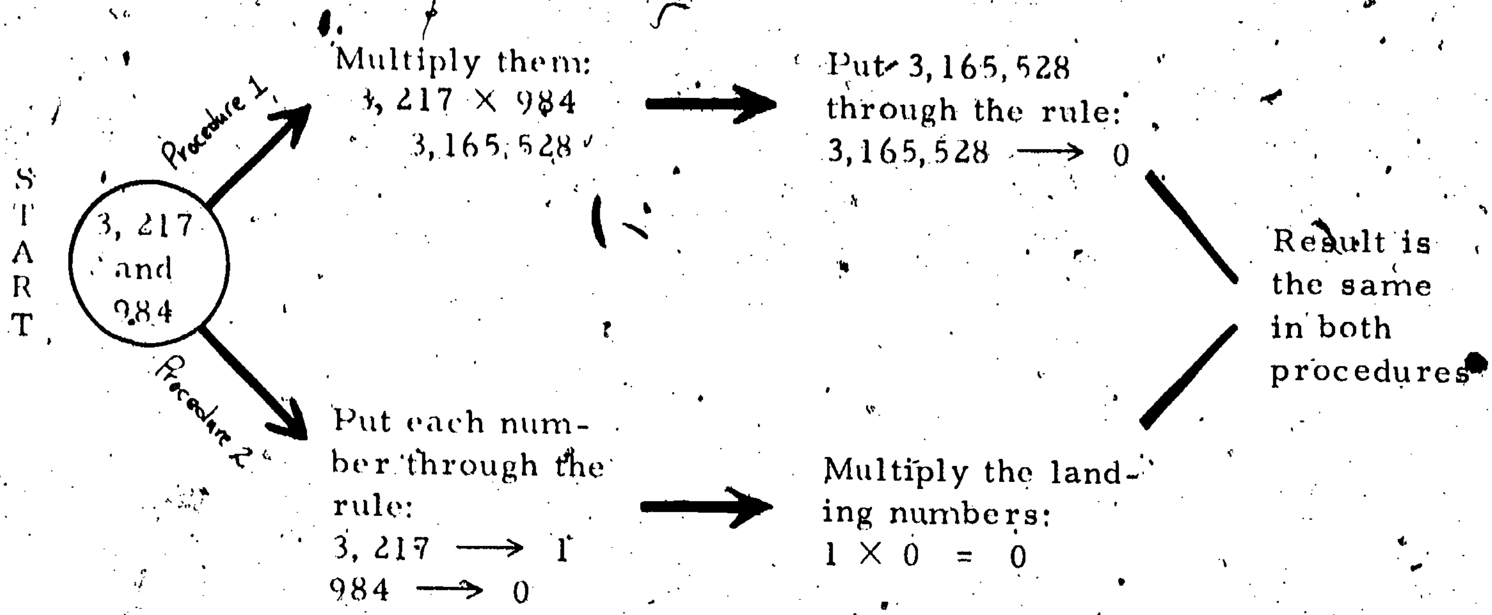
$$\square \longrightarrow \begin{cases} 0 & \text{if } \square \text{ is even} \\ 1 & \text{if } \square \text{ is odd} \end{cases}$$

(The legal starting numbers are all integers—0, 1, -1, 2, -2, . . .).

Choose any <u>two</u> starting numbers. We could follow two different procedures:

1. Multiply the numbers, and then put the product through the rule.
2. <u>First</u> put each number through the rule and then multiply the two landing numbers.

The rule has the remarkable property in that it makes no difference which procedure is followed.* For example, suppose the two numbers are 3,217 and 984. The procedures are diagramed below:



So if we're interested only in where we land if we start at the product of 3,217 and 984, we can avoid the bothersome multiplication by using Procedure 2 rather than Procedure 1.

The reader may see that this example can be summarized by saying that the product of two whole numbers is even unless both numbers are odd, in which case the product is odd. While this is of some theoretical interest, its practical value is negligible: if we are interested in finding what the product of 3,217 and 984 is, Procedure 2 is of no help. Even though we can find out where we land if we start at 3,217 × 984, we cannot use the landing number (0) to reconstruct the product. This is because there are many starting numbers (all the even numbers) which will give 0 when put through the rule.

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*If the reader doesn't find this remarkable, he is urged to find other functions with this property. One is □ ⟶ □ × □. What about □ ⟶ □ + 3 or □ ⟶ 5 × □ ?

Example XI.

$\Box \longrightarrow \left\{ \begin{array}{l} \text{the number of times 2} \\ \text{is used when } \Box \text{ is} \\ \text{written as a product of 2's} \end{array} \right.$

The legal starting numbers are those numbers which can be written as a product of 2's. Since $8 = 2 \times 2 \times 2$, if we start at 8 we land on 3. A partial list of jumps is shown below:

START		LAND
2	$\longrightarrow$	1
4	$\longrightarrow$	2
8	$\longrightarrow$	3
16	$\longrightarrow$	4
32	$\longrightarrow$	5
64	$\longrightarrow$	6
128	$\longrightarrow$	7
256	$\longrightarrow$	8

Pick any two starting numbers. Again we can follow two different procedures:

1. Multiply the numbers, and then put the product through the rule.

2. First put each number through the rule and then add the two landing numbers. Again the results of the two procedures will be the same. For example, suppose we start with 4 and 32.

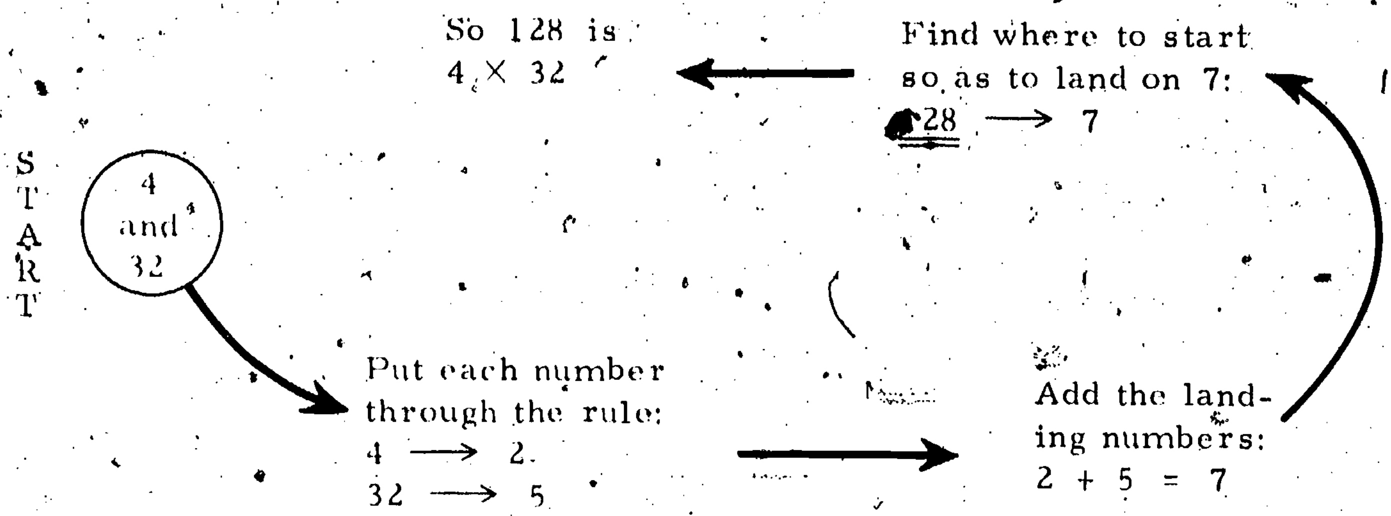
START: 4 and 32

Procedure 1: Multiply them: $4 \times 32 = 128$ $\longrightarrow$ Put 128 through the rule: $128 \longrightarrow 7$

Procedure 2: Put each number through the rule: $4 \longrightarrow 2$, $32 \longrightarrow 5$ $\longrightarrow$ Add the landing numbers: $2 + 5 = 7$

Result is the same in both procedures

It happens that 7 is the landing number for no other starting number except 128. Using this fact enables us to find the product of 4 and 32 in this fashion:



Thus the problem of multiplying two starting numbers is reduced to the far simpler problem of adding the landing numbers and then using the rule backwards. In the example above, instead of multiplying 4 and 32 we put 4 and 32 through the rule, landing on 2 and 5 respectively; we added the landing numbers, getting 7; since 128 is the place to start in order to land on 7 we know that $4 \times 32 = 128$.

At this point the reader may be thinking that this method works only if we are trying to multiply numbers like 2, 4, 8, 16, etc. If so, the reader is right; it would be silly to try to write 5, for example, as a product of 2's because $2 \times 2 = 4$ and $2 \times 2 \times 2 = 8$. But it would be a tremendous aid to computation if the rule could be extended to other starting points so that the same relationship between multiplying starting numbers and adding landing numbers held. Suppose we start at $\frac{1}{8}$. Where should we land? We know that $16 \times \frac{1}{8} = 2$. If we use these numbers as starting numbers then the landing numbers should give an equation involving addition:

$$\begin{array}{lccccc} \text{START:} & 16 & \times & \frac{1}{8} & = & 2 \\ & \downarrow & & \downarrow & & \downarrow \\ \text{LAND:} & 4 & + & ? & = & 1 \end{array}$$

But we know that $4 + (-3) = 1$, so if we start at $\frac{1}{8}$, we should land on -3. Notice that we are not claiming that if one uses 2 as a factor negative three times, the product is $\frac{1}{8}$. (How could you use anything negative three times?) We are claiming that if one wants to allow more numbers as legal starting places, keeping the same multiplication-addition idea, then in particular if we start at $\frac{1}{8}$ we must land on -3.

In the case of $16 \times \frac{1}{8} = 2$, it makes sense to say that if we start at $\frac{1}{8}$ we land on -3. But how do we know that this will always work? Perhaps another problem would force us to conclude that if we start at $\frac{1}{8}$ we must land on $-5\frac{1}{3}$, say. Of course this would be disastrous: it would mean that we could never include $\frac{1}{8}$ as a legal starting place since if our rule is to be a function every starting place must have exactly one landing place. Suppose S_1 and S_2 are any two legal starting numbers such that $S_1 \times \frac{1}{8} = S_2$. We will call L_1 the landing place for S_1, and L_2 the landing place for S_2. We want to show that $L_1 + (-3) = L_2$, for then we will have shown that the landing number for $\frac{1}{8}$ is -3. Now if $S_1 \times \frac{1}{8} = S_2$ then S_1 is 8 times as big as S_2, or, saying it a little differently, $S_2 \times 8 = S_1$. We know the landing places for all three of these numbers. We have:

$$\begin{array}{lccccc}
\text{START:} & S_2 & \times & 8 & = & S_1 \\
 & \downarrow & & \downarrow & & \downarrow \\
\text{LAND:} & L_2 & + & 3 & = & L_1
\end{array}$$

We know that the bottom equation is correct because S_2, 8 and S_1 are all legal starting numbers. But $L_2 + 3 = L_1$ says that L_1 is 3 more than L_2. So if we added -3 to L_1 we would have L_2. Thus $L_1 + (-3) = L_2$. We have shown that extending our rule to include $\frac{1}{8}$ as a legal starting number will cause no inconsistency.

Let's try to include a few more numbers as legal starting numbers for the rule. Where should we land if we start at 1? We have, for example:

$$\begin{array}{lccccc} \text{START:} & 16 & \times & 1 & = & 16 \\ & \downarrow & & \downarrow & & \downarrow \\ \text{LAND:} & 4 & + & ? & = & 4 \end{array}$$

But $4 + 0 = 4$, so if we start at 1 we must land on 0. Again we could check this to see that making 0 the landing number of 1 produces no contradiction. In somewhat fancier language, if multiplication problems involving starting numbers are going to be turned into addition problems involving the landing numbers, then if we start at the identity element for multiplication (namely 1) we should land on the identity element for addition (namely 0).

Where should we land if we start at 0? Using the same techniques as before we have for example:

$$\begin{array}{lccccc} \text{START:} & 8 & \times & 0 & = & 0 \\ & \downarrow & & \downarrow & & \downarrow \\ \text{LAND:} & 3 & + & \triangle & = & \triangle \end{array}$$

Notice that we have used two frames <u>of the same shape</u> in the second equation because starting at zero must give us exactly <u>one</u> landing number. But it is impossible to fill the blanks of $3 + \triangle = \triangle$ to get a true sentence. Therefore we <u>cannot</u> extend our rule to include 0 as a legal starting number.

Where should we land if we start at 5 ? We notice that $5 \times 5 \times 5 = 125$, and 125 is pretty close to 128 . So,

START: $5 \times 5 \times 5 = 125$ close to 128

LAND: $\triangle + \triangle + \triangle =$ close to 7

If $\triangle + \triangle + \triangle$ is to be near 7 then $\triangle$ should be close to $\frac{7}{3}$ or $2\frac{1}{3}$. Using more powerful methods which we will not explain here, one finds that to five decimal places the landing number is 2.32192, so our estimate of $2\frac{1}{3}$ is not too far off.

It turns out that this function (which is technically known as a logarithm function) can be extended to include all positive numbers as legal starting places. The whole idea was first developed by John Napier and Henry Briggs in the early seventeenth century, and a complete table of landing numbers was published in 1627. It vastly simplified the horrendous calculations necessary for the astronomical, navigational and engineering applications of the day. The logarithm function (usually in a slightly different form) is used now in many practical and theoretical settings. Its usefulness lies in two of its properties: First, as we have seen, multiplication problems are turned into addition problems, and, second, each landing number is the landing number for only one starting number. This allows us to reconstruct uniquely the product from the sum by using the logarithm function backwards.

* * *

It should be emphasized that this paper is only an introduction to the nature and uses of functions. But elementary school teachers should be assured that there are few topics which play a more central role in every branch of mathematics.